The Energy Cascade in Strong Wave Turbulence

Wave turbulence concerns the statistical description of ensembles of interacting dispersive waves maintained far from equilibrium by coupling the wave field to external sources and sinks of energy. The archetypal example is gravity-capillary waves on the surface of water driven by the wind and damped by viscosity. Other applications include optical waves of diffraction in nonlinear media, spin waves in magnetic materials and Alfven waves in magnetized plasmas to mention a few. Aside from nonlinear interactions between waves, the other essential ingredient to generate wave turbulence is a wide separation in scale between the sources and sinks of energy. Just as in the case of hydrodynamic turbulence, this leads to the formation of an inertial range of scales through which energy is conservatively transferred by the nonlinear interactions from the source scale to the sink. This process is known as an energy cascade. By carrying a flux of energy through the inertial range, a cascade allows a wave field to reach a non-equilibrium stationary state where forcing balances dissipation on average. In reality, other quantities may also produce cascades but here we only consider energy cascades.

We consider systems where the leading nonlinearity is quadratic (3-wave turbulence). Capillary waves and Alfven waves are in this class. Conservative dynamics admit a natural Hamiltonian description in terms of Fourier complex canonical variables, $\{a_{\mathbf{k}}, \bar{a}_{\mathbf{k}}\}$,

$$H = T + U = \int d\mathbf{k}\ \omega_{\mathbf{k}} \bar{a}_{\mathbf{k}} a_{\mathbf{k}} + \int d\mathbf{k}\ u(\mathbf{k}). \qquad (1)$$

ω is the linear frequency and $u(\mathbf{k})$ is the nonlinear (interacting) part of the energy density. When $u = 0$, Eq. (1) describes free waves with dispersion law $\omega_{\mathbf{k}}$, $\mathbf{k}$ being the wave vector. We now supplement Hamilton's equations with forcing and dissipation giving the equations of motion

$$\partial_t a_{\mathbf{k}} = i\omega_{\mathbf{k}} a_{\mathbf{k}} + i\frac{\delta U}{\delta \bar{a}_{\mathbf{k}}} + f_{\mathbf{k}} - \nu_{\mathbf{k}} a_{\mathbf{k}}. \qquad (2)$$

For 3-wave turbulence, the nonlinear energy density takes the general form:

$$u = \int d\mathbf{k}_1 d\mathbf{k}_2 \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) T_{\mathbf{k};\mathbf{k}_1,\mathbf{k}_2} [\bar{a}_{\mathbf{k}} a_{\mathbf{k}_1} a_{\mathbf{k}_2} + \text{c.c}].$$

In this formulation, all details of the physics is contained in the form of the dispersion relation, $\omega_{\mathbf{k}}$, and the interaction coefficient, $T_{\mathbf{k};\mathbf{k}_1,\mathbf{k}_2}$. In general, these are very complicated functions. Nevertheless, in many applications they are homogeneous functions of their arguments reflecting an underlying scale invariance of the original physical system. We denote by α and γ the degrees of homogeneity of $\omega_{\mathbf{k}}$ and $T_{\mathbf{k};\mathbf{k}_1,\mathbf{k}_2}$ respectively. Together with the physical dimension of the wave field, d, the exponents α and γ determine many of the scaling properties of the wave turbulence described by Eq. (2).

In the inertial range, $f_{\mathbf{k}}$ and $\nu_{\mathbf{k}}$ are negligible and the dynamics are entirely controlled by the Hamiltonian part. In the case of weak interactions between waves, so called weak wave turbulence, an essentially complete theory was developed by Hasselmann, Newell, Zakharov and others in the '60s and '70s. In weak wave turbulence, the principle contribution to H is the quadratic energy T. Nonlinear interactions localise on a set of resonant manifolds and a consistent kinetic theory can be developed which describes the redistribution of T by wave resonances. Higher order moments of the wave field are expressed in terms of the second moment, $\langle \bar{a}_{\mathbf{k}_1} a_{\mathbf{k}_2} \rangle = n_{\mathbf{k}_1} \delta(\mathbf{k}_1 - \mathbf{k}_2)$. The kinetic equation can be exactly solved for the stationary state using a technique now known as the Zakharov Transformation. The spectrum, $n_{\mathbf{k}} = c\sqrt{\varepsilon}\, k^{-\gamma-d}$ is an exact stationary solution of the kinetic equation which carries a constant flux, ε, of quadratic energy, from large scales to small. The constant c is calculable. This spectrum is the analogue of the Kolmogorov spectrum of fluid turbulence. Much work on wave turbulence has focused on its verification.

Despite an extensive theoretical framework for understanding weak interactions, relatively little is known about the case where interactions between waves are strong. The lack of a consistent closure for higher moments of the theory provides a formidable obstacle. Nevertheless, we argue that the flux carrying correlation function may be determined exactly, even in the case where wave interactions are strong. This is by analogy with the derivation of the $4/5$ law in hydrodynamic turbulence and does not require any

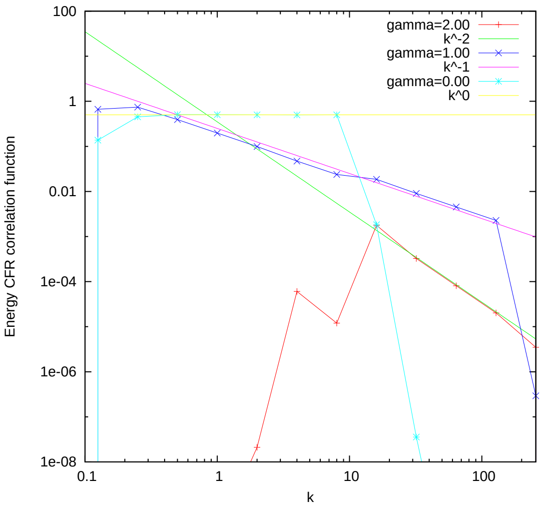

Figure 1: Flux-correlation function of the model system, Eq. (4), in the stationary state for several values of γ compared with predicted scaling.

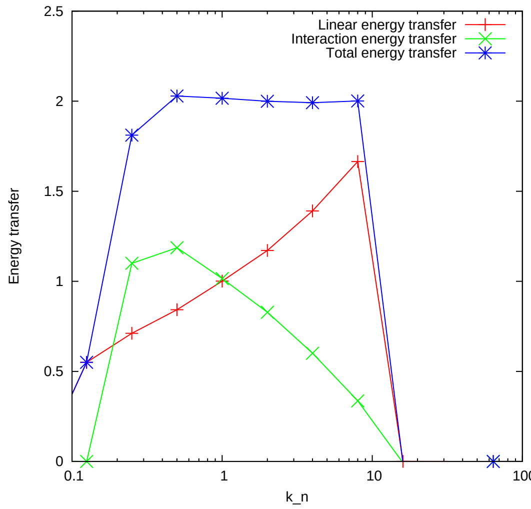

Figure 2: Conversion of linear to nonlinear energy gives a constant energy flux in the model, Eq. (4)

closure assumptions. One difficulty in treating the strongly interacting case is that, in deriving the flux equation, one must take into account that it is the total energy, H, which is conserved by the nonlinear interactions. With this in mind, one can show that the following continuity equation holds in the inertial range in the stationary state:

$$\int \prod_{i=1}^{2} (dk_i k_i^{d-1}) \left[T_{\mathbf{k};\mathbf{k}_1,\mathbf{k}_2} \Pi_{0;1,2} - T_{\mathbf{k}_1;\mathbf{k},\mathbf{k}_2} \Pi_{1;0,2} \right] = 0, \tag{3}$$

where $\Pi_{0;1,2} = \int \prod_{i=0}^{2} d\Omega_i \langle Re(a_{\mathbf{k}}\ \partial_t \bar{a}_{\mathbf{k}_1}\ \bar{a}_{\mathbf{k}_2}) \rangle$ is the flux correlation function. Since it involves correlations of fields and time derivatives of fields, it is a new type of object which has not arisen previously in systems with quadratic invariants. If we assume that energy transfer is local in scale and that $\Pi_{0;1,2}$ is a homogeneous function of degree, h, one can use the Zakharov Transformation to show that the exponent $h = -\gamma - 3d$ exactly solves Eq. (3). This scaling for $\Pi_{0;1,2}$ describes a constant flux of total energy, $T + U$, in the inertial range. This scaling should hold for both weak and strong wave turbulence.

Even if local transfer is assured, which is difficult to verify in general, it is strange that an object like $\langle Re(a_{\vec{k}} \dot{\bar{a}}_{\vec{k}_1} \bar{a}_{\vec{k}_2}) \rangle$ should scale since it can be expressed as a sum of correlation functions of different orders. To demonstrate proof of concept, we studied the model Hamiltonian:

$$H = \sum_{n=-N}^{n=N} \omega_n a_n \bar{a}_n + k_{n-1}^{\gamma} \left(\bar{a}_n a_{n-1}^2 + a_n \bar{a}_{n-1}^2 \right) \tag{4}$$

where $k_n = 2^n$. Locality is enforced since the Hamiltonian only contains the interactions $k_n + k_n \rightarrow k_{n+1}$. The statement that this model system has a constant flux of total energy can be expressed as

$$\mathrm{Re}\langle \bar{a}_{n+1}\ a_n\ d_t a_n \rangle = -\frac{Q_0^{(H)}}{4}\ k_n^{-\gamma}, \tag{5}$$

again involving correlations of fields and derivatives. Here $Q_0^{(H)}$ is the flux. A numerical validation of this relation is shown in Fig. 1 for a sequence of values of the parameter γ. An illustration of how the cascade works is shown in Fig. 2. Neither the flux of linear energy, T, nor nonlinear energy, U, are constant, but by converting T into U, a constant flux of total energy, $T + U$ can be obtained. We are now attempting to find if such a mechanism can produce a cascade of total energy in a more realistic model of wave turbulence where locality of interaction is not assured a-priori. If so, this work will provide the first concrete, generally applicable theoretical handle on strong wave turbulence.

References

[1] C. Connaughton, R. Rajesh, and O. Zaboronski. Constant Flux Relation for driven dissipative systems. *Phys. Rev. Lett.*, 98:080601, 2007, cond-mat/0607656.

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